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NEW PARABOLIZED SYSTEM OF EQUATIONS OF STABILITY OF A COMPRESSIBLE BOUNDARY LAYER

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Based on estimates for the critical layer, a system of equations of stability of a compressible boundary layer is obtained. The system is parabolic and free from the known restriction on the step of the marching scheme related to ellipticity, which could not be eliminated within the framework of the previous method. A numerical scheme is described, and calculation results for the boundary layer on a heat-insulated plate are presented.

In deriving parabolized equations of boundary-layer stability (see, for example, [1]), the authors used previously asymptotic estimates for integer powers of $R = \sqrt{\mathrm{Re}}$, where Re is the Reynolds number based on the distance x from the plate edge. Terms of order R^{-1} are retained in the equations (here and in what follows, estimates of the terms of the equations are taken relative to the main terms). It is known, however, that perturbations of velocity and viscous stresses reach the highest values in the critical layer in which different estimates including fractional powers of R are valid. Dunn and Lin [2] ignore terms of order R^{-1} even outside the critical and near-wall layers, thus, assuming that the equations of the inviscid parallel theory of stability are rather accurate. Among the equations for the critical layer, Dunn and Lin [2] took into account only the main terms, whereas the preliminary estimate for the term containing the x derivative of the amplitude function of the streamwise component of velocity perturbation yields the order $R^{-1/3}$. In the present paper, we ignore only terms of order R^{-1} that contain perturbations of viscous stresses or x derivatives of the amplitude functions of the perturbations.

The equations of dynamics of a viscous compressible fluid in an arbitrary orthogonal coordinate system (ξ_1, ξ_2, ξ_3) [3] are represented in the form

$$d_t\rho + \rho \operatorname{div} \boldsymbol{v} = 0,$$

$$\rho d_t v_i + \sum_{\substack{m=1\\ m\neq i}}^{3} [H_{im}(\rho v_i v_m - \tau_{im}) - H_{mi}(\rho v_m^2 - \tau_{mm})] = -\partial_i p + \operatorname{div} \boldsymbol{\tau}_i \qquad (i = 1,\, 2,\, 3),$$

$$\rho d_t H = \partial_t p + \operatorname{div} \boldsymbol{q},$$

where $\boldsymbol{v}$ is the velocity, ρ is the density, p is the pressure, and H is the total enthalpy,

$$d_t = \partial_t + \sum_{k=1}^{3} v_k \partial_k, \quad \partial_t = \frac{\partial}{\partial t}, \quad \partial_k = \frac{1}{H_k}\frac{\partial}{\partial \xi_k}, \quad \operatorname{div} \boldsymbol{a} = \sum_{k=1}^{3}(\partial_k + h_k)a_k,$$

$$h_k = \sum_{\substack{m=1\\ m\neq k}}^{3} H_{mk}, \quad H_{mk} = \partial_k \ln H_m, \quad H_m = \sqrt{\Big(\frac{\partial x}{\partial \xi_m}\Big)^2 + \Big(\frac{\partial y}{\partial \xi_m}\Big)^2 + \Big(\frac{\partial z}{\partial \xi_m}\Big)^2},$$

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$$\tau_{ik} = \mu(\partial_k v_i + \partial_i v_k - H_{ik}v_i - H_{ki}v_k), \qquad \tau_{ii} = 2\mu\biggl(\partial_i v_i + \sum_{\substack{m=1\\ m\neq i}}^{3} H_{im}v_m - \frac{1}{3}\operatorname{div}\boldsymbol{v}\biggr),$$

$\boldsymbol{\tau}_i = (\tau_{i1}, \tau_{i2}, \tau_{i3})$, $q_k = \lambda\partial_k T + \boldsymbol{\tau}_k\boldsymbol{v}$, t is the time, x, y, and z are the Cartesian coordinates, T is the temperature, μ is the viscosity, and λ is the thermal conductivity.

The linearized system of equations for a perturbation $\hat{a}$ of a flow parameter a ($a \Rightarrow a + \hat{a}$) has the form

$$(d_t + \operatorname{div}\boldsymbol{v})\hat{\rho} + \sum_{k=1}^{3}(\partial_k\rho)\hat{v}_k + \rho\operatorname{div}\hat{\boldsymbol{v}} = 0,$$

$$\rho\biggl[d_t\hat{v}_i + \sum_{k=1}^{3}(\partial_k v_i)\hat{v}_k\biggr] + \biggl(\sum_{k=1}^{3} v_k\partial_k v_i\biggr)\hat{\rho} + \sum_{\substack{m=1\\ m\neq i}}^{3}[H_{im}(v_i v_m\hat{\rho} + \rho v_i\hat{v}_m + \rho v_m\hat{v}_i - \hat{\tau}_{im})$$

$$- H_{mi}(v_m^2\hat{\rho} + 2\rho v_m\hat{v}_m - \hat{\tau}_{mm})] = -\partial_i\hat{p} + \operatorname{div}\hat{\boldsymbol{\tau}}_i,$$

$$\hat{\tau}_{ik} = \mu[(\partial_k - H_{ik})\hat{v}_i + (\partial_i - H_{ki})\hat{v}_k] + \tau_{ik}\frac{\hat{\mu}}{\mu}, \quad \hat{\tau}_{ii} = 2\mu\biggl(\partial_i\hat{v}_i + \sum_{\substack{m=1\\ m\neq i}}^{3} H_{im}\hat{v}_m - \frac{1}{3}\operatorname{div}\hat{\boldsymbol{v}}\biggr) + \tau_{ii}\frac{\hat{\mu}}{\mu},$$

$$\rho d_t\hat{H} + \sum_{k=1}^{3}(\partial_k H)(\rho\hat{v}_k + v_k\hat{\rho}) = \partial_t\hat{p} + \operatorname{div}\hat{\boldsymbol{q}}, \qquad \hat{q}_k = \lambda\partial_k\hat{T} + (\partial_k T)\hat{\lambda} + \boldsymbol{\tau}_k\hat{\boldsymbol{v}} + \boldsymbol{v}\hat{\boldsymbol{\tau}}_k.$$

Below, we consider a planar problem using the coordinate system (ξ, ψ), where ψ is the stream function of the main flow. Then, we have $H_2 = 1/(\rho u)$, $h_1 = -\partial_1\ln(\rho u)$, $h_2 = \partial_2\ln H_1$ is the streamline curvature, and u is the steady velocity. Naturally, we assume $\psi = 0$ on the wall and determine the Lamé coefficient H_1 through the curvature by the integral $H_1 = \exp\int_0^{\psi} h_2 H_2\,d\psi$; the value of ξ on the wall is the distance along it.

We consider a perturbation in the form of a monochromatic wave with angular frequency ω:

$$\hat{a}(\xi, \psi, t) = \tilde{a}(\xi, \psi)\exp\Bigl\{i\Bigl[\int k(\xi)\,d\xi - \omega t\Bigr]\Bigr\}.$$

Substituting the notation of the amplitude functions of the velocity-perturbation components $\hat{v}_1$ and $\hat{v}_2$ (along the streamlines and normal to them) with $\tilde{u}$ and $\tilde{v}$, we obtain

$$(\partial_2 + h_2)\tilde{v} + (h_1 + i\alpha + \partial_1)\tilde{u} - \tilde{e} = 0,$$

$$\partial_2(\tilde{p} - \tilde{\tau}_{22}) + \rho(h_1 u + d_t)\tilde{v} - h_2 u(2\rho\tilde{u} + u\tilde{\rho}) - i\alpha\tilde{\tau}_{12} = (2h_1 + \partial_1)\tilde{\tau}_{12} + h_2(\tilde{\tau}_{22} - \tilde{\tau}_{11}),$$

$$(\partial_2 + 2h_2)\tilde{\tau}_{12} - (i\alpha + \partial_1)\tilde{p} - \rho(\partial_2 u + h_2 u)\tilde{v} - \rho(\partial_1 u + d_t)\tilde{u} - (u\partial_1 u)\tilde{\rho} + i\alpha\tilde{\tau}_{11} = -(h_1 + \partial_1)\tilde{\tau}_{11} + h_1\tilde{\tau}_{22},$$

$$(\partial_2 - h_2)\tilde{u} + (i\alpha - h_1)\tilde{v} + \tau_{12}\tilde{\mu}/\mu^2 - \tilde{\tau}_{12}/\mu = -\partial_1\tilde{v}, \tag{1}$$

$$(\partial_2 + h_2)\tilde{q}_2 - i\omega\tilde{p} - (\rho\partial_2 H)\tilde{v} - (\rho\partial_1 H)\tilde{u} - \rho d_t\tilde{H} - (u\partial_1 H)\tilde{\rho} + i\alpha\tilde{q}_1 = -(h_1 + \partial_1)\tilde{q}_1,$$

$$\lambda\partial_2\tilde{T} + \tau_{12}\tilde{u} + u\tilde{\tau}_{12} + (\partial_2 T)\tilde{\lambda} - \tilde{q}_2 = -\tau_{22}\tilde{v},$$

$$\tilde{\tau}_{11} - 2\mu(i\alpha\tilde{u} - \tilde{e}/3) = 2\mu(h_2\tilde{v} + \partial_1\tilde{u}) + \tau_{11}\tilde{\mu}/\mu, \quad \tilde{\tau}_{22} - 2\mu(\partial_2\tilde{v} - \tilde{e}/3) = 2\mu h_1\tilde{u} + \tau_{22}\tilde{\mu}/\mu,$$

$$\tilde{q}_1 - \tau_{12}\tilde{v} - i\alpha\lambda\tilde{T} - u\tilde{\tau}_{11} = \tau_{11}\tilde{u} + \lambda\partial_1\tilde{T} + (\partial_1 T)\tilde{\lambda}, \quad \rho\tilde{e} = -(\partial_2\rho)\tilde{v} - (\partial_1\rho)\tilde{u} - (e + d_t)\tilde{\rho},$$

TABLE 1

Power Indices b Determining the Order of $O(R^b)$ of Parameters and Operators for Perturbations in the Critical Layer

Order of	Parameters and operators					
α, ω	δ_c, $\tilde{p}$	∂_1, $\tilde{\tau}_{12}$, $\tilde{q}_2$	∂_2	$\tilde{\tau}_{11}$, $\tilde{\tau}_{22}$, $\tilde{q}_1$	$\tilde{v}$, $\tilde{e}$, u_c	$\tilde{l}_k$
$O(1)$	$-1/3$	$-2/3$	$1/3$	-1	$-1/3$	-1
$O(R^{-1/2})$	$-1/6$	$-5/6$	$1/6$	$-3/2$	$-2/3$	$-4/3$
$O(R^{-1})$	0	-1	0	-2	-1	-2

where $d_t = u_c + u\partial_1$, $u_c = i(u\alpha - \omega)$, $\alpha = k/H_1$, $e = (\partial_1 + h_1)u$, $\tau_{12} = \mu(\partial_2 - h_2)u$, $\tau_{11} = 2\mu(\partial_1 u - e/3)$, and $\tau_{22} = 2\mu(h_1 u - e/3)$.

By analogy with [2], we evaluate the terms of system (1) for $R \to \infty$ in the critical layer of thickness δ_c located in the vicinity of the point $u_c = 0$. We use the boundary-layer thickness δ as a length scale, $R = u_e\delta/\nu_e$, where $\nu_e = \mu_e/\rho_e$, and the subscript e corresponds to the value at the boundary-layer edge. Then, we have $\xi = O(R)$; for the main flow, $\partial_1 = O(R^{-1})$ and $\partial_2 = O(1)$ (∂_1 and ∂_2 are roughly equal to derivatives with respect to the usual coordinates of the boundary layer), and $\mu, \lambda = O(R^{-1})$. The viscous ($\mu\partial_2^2\tilde{u}$) and inviscid ($\rho u_c\tilde{u}$) terms in the second-order equations equivalent to the third and fourth equations of system (1) should be of the same order. With account of the estimate $u_c = O(\alpha\delta_c)$, it follows that $\delta_c = O((\alpha R)^{-1/3})$.

It is assumed [2] that $\alpha = O(1)$; then we have $\delta = O(R^{-1/3})$. This is valid for higher modes, but for neutral perturbations of the fundamental mode we have $\alpha = O(R^{-1})$. The power indices b determining the order $O(R^b)$ of the parameters and operators for perturbations in the critical layer are listed in Table 1. Apart from the extreme cases $\alpha = O(1)$ and $\alpha = O(R^{-1})$, Table 1 gives data for an intermediate case $\alpha = O(R^{-1/2})$. We assume (the linear perturbation is determined with accuracy to an arbitrary factor) that $\tilde{u}$, $\tilde{H}$, $\tilde{T}$, and $\tilde{\rho}$ are of order $O(1)$. Estimates for $\tilde{v}$ and $\tilde{p}$ are obtained from the requirement that they should be contained in the main parts of the first and third equations of system (1).

The right sides of the first six equations consist of terms of order R^b ($b < -1$) and, therefore, are rejected. In the next three equations determining $\tilde{\tau}_{11}$, $\tilde{\tau}_{22}$, and $\tilde{q}_1$, it is also sufficient to retain only the left parts for the previous equations to contain all terms of order R^{-1}. Using the approximate relations $\partial_2\tilde{v} = -i\alpha\tilde{u} - (\partial_2 \ln \rho)\tilde{v} - (u_c/\rho)\tilde{\rho}$ and $\partial_2\tilde{u} = \tilde{\tau}_{12}/\mu - \tau_{12}\tilde{\mu}/\mu^2$, we obtain

$$\begin{gathered}
\partial_2\tilde{v} = -(\partial_2 \ln \rho + h_2)\tilde{v} + (\partial_1 \ln u - i\alpha - \partial_1)\tilde{u} - (1/\rho)(u_c - u\partial_1 \ln \rho + u\partial_1)\tilde{\rho}, \\
\partial_2\tilde{p} = -\rho(u_c + h_1 u)\tilde{v} + h_2 u(2\rho\tilde{u} + u\tilde{\rho}) - i\alpha\tilde{\tau} - \tilde{l}_1 + \tilde{l}_2, \\
\partial_2\tilde{\tau} = (i\alpha + \partial_1)\tilde{p} + \rho(\partial_2 u + h_2 u)\tilde{v} + (\rho\partial_1 u + 2\alpha^2\mu + \rho u_c + \rho u\partial_1)\tilde{u} + (u\partial_1 u)\tilde{\rho} - 2h_2\tilde{\tau} - \tilde{l}_3, \\
\partial_2\tilde{u} = (h_1 - i\alpha)\tilde{v} + h_2\tilde{u} - (\partial_2 u/\mu)\tilde{\mu} + \tilde{\tau}/\mu, \\
\partial_2\tilde{q} = i\omega\tilde{p} + (\rho\partial_2 H)\tilde{v} + (\rho\partial_1 H + 2\alpha^2\mu u)\tilde{u} + \rho(u_c + u\partial_1)\tilde{H} + \alpha^2\lambda\tilde{T} + (u\partial_1 H)\tilde{\rho} - h_2\tilde{q} - \tilde{l}_4, \\
\lambda\partial_2\tilde{T} = -(\mu\partial_2 u)\tilde{u} - u\tilde{\tau} - (\partial_2 T)\tilde{\lambda} + \tilde{q},
\end{gathered} \tag{2}$$

where $\tilde{\tau}$ and $\tilde{q}$ correspond to $\tilde{\tau}_{12}$ and $\tilde{q}_2$ in (1),

$$\begin{gathered}
\tilde{l}_1 = \rho u\partial_1\tilde{v}; \qquad \tilde{l}_2 = 2i\alpha[(\partial_2\mu - 2\mu\partial_2 \ln(\rho)/3)\tilde{u} - (\partial_2 u)\tilde{\mu}] + (4\mu u_c/(3\rho))\partial_2\tilde{\rho}, \\
\tilde{l}_3 = (2i\alpha\mu/3)[(\partial_2 \ln \rho)\tilde{v} + (u_c/\rho)\tilde{\rho}]; \qquad \tilde{l}_4 = u\tilde{l}_3 + i\alpha\mu(\partial_2 u)\tilde{v}.
\end{gathered} \tag{3}$$

It follows from the estimates given in Table 1 that $\tilde{l}_k$ reach the order R^{-1}. These terms contain only perturbations of viscous stresses or ξ derivatives of the amplitude functions and, in accordance with the above assumptions, should be rejected:

$$\tilde{l}_k = 0, \qquad k = 1\text{–}4. \tag{4}$$

System (2), (4) does not contain $\partial\tilde{v}/\partial\xi$ and, with account of linearized equations of state of the gas, can be reduced to the canonical parabolized form $\partial Z/\partial\xi = A\partial^2 Z/\partial\psi^2 + B$ [$Z = (\tilde{p}, \tilde{u}, \tilde{H})$, A is the matrix of functions of coordinates, and B is the vector function of ξ, ψ, Z, and $\partial Z/\partial\psi$]; therefore, it is free from ellipticity limiting the step $\Delta\xi$ of stable calculation by implicit marching schemes [1] by the inequality

$$|\alpha|\Delta\xi > 1. \tag{5}$$

Parabolicity is retained after nontensor transformations of the form $\xi_1 = \xi_1(\xi)$ and $\xi_2 = \xi_2(\xi, \psi)$, for example, in passing to a variable of similarity for the boundary layer on a flat plate.

The terms taking into account the curvature of streamlines can be excluded if (on a scale of δ) $h_2 R \ll 1$. With account of $H_1 = 1 + O(h_2)$, the relations $\partial_1 = \partial/\partial\xi$ and $\alpha = k$ are rather accurate. For the flow past a flat plate considered below, we have $h_2 = O(R^{-2})$.

We use the following scales: ν_e/u_e for distance, u_e for velocity and components of its perturbation, u_e/ν_e for ∂_1, ∂_2, and α, u_e^2/ν_e for ω and u_c, μ_e for ψ, μ, and $\tilde{\mu}$, u_e^2 for H and $\tilde{H}$, ρ_e for ρ and $\tilde{\rho}$, $\rho_e u_e^2$ for $\tilde{p}$ and $\tilde{\tau}$, $\rho_e u_e^3$ for $\tilde{q}$, T_e for T and $\tilde{T}$, and $\mu_e u_e^2/T_e$ for λ and $\tilde{\lambda}$. The quantities with the subscript e are constant; for a flat plate, these are the values at the boundary-layer edge. In these dimensions, the form of Eqs. (2) and (3) does not change.

We pass to the variable of similarity $f = \psi/\sqrt{\xi}$, and then to $R = \sqrt{\xi}$, and $d\eta = df/u$, so that

$$\partial_1 = \frac{1}{2R}\Big(\frac{\partial}{\partial R} - \frac{f}{Ru}\frac{\partial}{\partial\eta}\Big), \qquad \partial_2 = \frac{\rho}{R}\frac{\partial}{\partial\eta}.$$

For a perfect gas with a constant Prandtl number Pr, the following relations are valid: $\lambda = \mu/(\mathrm{Pr}\, g_{m1})$, $T = g_{m1}h$, $\rho = 1/T$, $\tilde{\rho}/\rho = g_m\tilde{p} - \tilde{T}/T$, $g_m = \gamma\mathrm{M}^2$, and $g_{m1} = (\gamma - 1)\,\mathrm{M}^2$ (M is the free-stream Mach number and $\gamma = c_p/c_v$ is the ratio of specific heats). Equations (2) and (4) take the form

$$\tilde{v}' = \rho T'\tilde{v} - (i_0 + f_0 T u' + \partial)\tilde{u} - T^2 i_c\tilde{p} - g_m u\partial\tilde{p} + \rho(-f_2 T' + u\partial)\tilde{T} - f_1 T\tilde{u}' + f_2\tilde{T}',$$

$$\tilde{p}' = -(i_c + r_h u)\tilde{v} - i_0\tilde{\tau}, \quad \tilde{\tau}' = (i_0 + \partial)\tilde{p} + \rho u'\tilde{v} + (i_c + f_1 u' + 2a_t + \rho u\partial)\tilde{u} + f_2 u' T\tilde{p} + f_2\tilde{u}',$$

$$\tilde{u}' = (r_h T - i_0)\tilde{v} - u'\tilde{\mu}/\mu + \tilde{\tau}/\mu_0, \tag{6}$$

$$\tilde{q}' = i\omega RT\tilde{p} + \rho H'\tilde{v} + (f_1 H' + a_t u)\tilde{u} + f_1 H' u T\tilde{p} + (i_c + a_t + \rho u\partial)\tilde{H} + a_t(1/\mathrm{Pr} - 1)\tilde{h} + f_2\tilde{H}',$$

$$\tilde{h}' = -\mathrm{Pr}\, u'\tilde{u} - h'\tilde{\mu}/\mu + (\mathrm{Pr}/\mu_0)(\tilde{q} - u\tilde{\tau}), \qquad \tilde{T} = g_{m1}\tilde{h}, \qquad \tilde{H} = \tilde{h} + u\tilde{u},$$

where $\partial = RT(\partial/\partial\xi) = (T/2)(\partial/\partial R)$, $i_0 = i\alpha RT$, $a_t = \alpha^2 RT\mu$, $i_c = Ru_c = iR(u\alpha - \omega)$, $\mu_0 = \mu/(RT)$, $f_0 = f/(2u^2R)$, $f_1 = -f_0 u$, $f_2 = f_1 u$, and $r_h = Rh_1 = f_0 u' + f_1\rho T'$; the prime indicates derivatives with respect to η.

Using in Eqs. (6) a simple approximation of derivatives with respect to R $\partial\tilde{a}/\partial R \approx (\tilde{a} - \tilde{a}_0)/\Delta R$, where $\Delta R = R - R_0$ is the step of the marching scheme, we obtain a system of ordinary differential equations. With account of obvious substitutions, this system acquires the form $Z' = AZ + B(Z - Z_0)$, where $Z = (\tilde{v}, \tilde{p}, \tilde{\tau}, \tilde{u}, \tilde{q}, \tilde{H})$ and A and B are matrices composed of the coefficients of Eqs. (6). Hereinafter, the subscript 0 indicates a quantity calculated at the previous step with respect to ξ. For a given α, the general solution of this system is the superposition $Z = \sum_{k=1}^{6} C_k Z_k + Z_n$ of linearly independent solutions of the homogeneous system $Z' = (A + B)Z$ and an arbitrary solution of the inhomogeneous system.

To derive the conditions at the outer edge of the boundary layer, we get back to Eqs. (2) and (4) using the previous notation for the corresponding system of ordinary differential equations. The neglect of terms

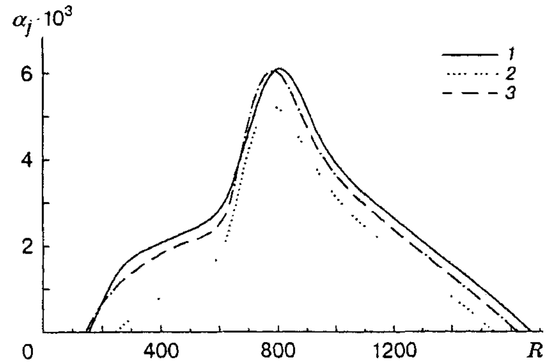

Fig. 1

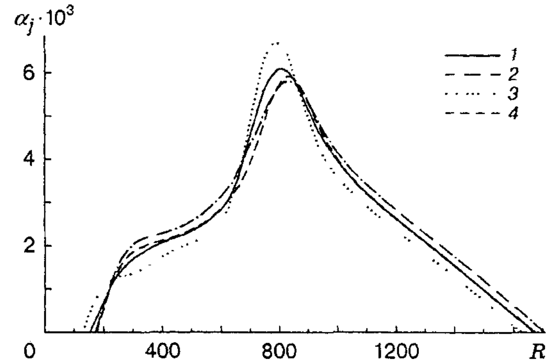

Fig. 2

containing $\partial_1 \tilde{a}$ outside the boundary layer means that $B = 0$ for $\psi > \psi_e$, i.e., the system is homogeneous. In the case of the flow around a flat plate, the general solution is a superposition of three decaying and three growing perturbations for $\psi \to \infty$ with pre-exponents Z_{ke} known from parallel theory. The solution in the boundary-layer region corresponding to the requirement of perturbation decay outside its boundaries can be constructed from four vectors using the following boundary conditions for them:

$$Z_n = CZ_{3e}, \qquad Z_k = Z_{ke}, \qquad k = 1\text{–}3, \qquad \eta = \eta_e. \tag{7}$$

The inviscid vector Z_{3e} is used in the condition for the inhomogeneous equation but it is possible to use an arbitrary superposition Z_{ke}.

Four vectors are determined by simultaneous numerical integration of the corresponding equations from the outer edge of the boundary layer to the wall. In doing so, orthogonalizations are used (Z_n is the last term to be orthogonalized; it is not orthogonalized relative to Z_3 and is not normalized). The constants C_k are determined by the conditions on the wall $\tilde{v} = \tilde{u} = \tilde{H} = 0$ for $\eta = 0$.

Introduction of an unknown function $\alpha(\xi)$ makes system (1) nonlinear and indeterminate. The problem can be made determinate, for example, using the conditions of a constant amplitude function of an arbitrary physical quantity at the line of maximum amplitude of mass-flow fluctuations $\tilde{m} = \rho\tilde{u} + u\tilde{\rho}$. This line corresponds to the critical layer as $R \to \infty$.

We used Newton's method to calculate α. The iteration process is directed to fulfill one of two conditions

$$\tilde{m} - \tilde{m}_0 = 0 \quad \text{for} \quad \eta = \eta_{\max}, \tag{8}$$

$$\tilde{p} - \tilde{p}_0 = 0 \quad \text{for} \quad \eta = \eta_{\max}, \tag{9}$$

where $\eta_{\max}$ is the maximum of the function $|\tilde{m}_0(\eta)|$. The same conditions are applicable for the flow around an arbitrary body, where η is understood as an arbitrary coordinate chosen for boundary-layer calculations.

If we use the coordinate system (ξ, ψ), condition (9) corresponds to the condition $\partial_1 \tilde{p} = 0$ for $\psi = \psi_{\max}$. Differentiating the second equation from (2) with respect to ξ and confining ourselves to the case of an ordinary boundary layer where the curvature h_2 in this equation can be ignored, we obtain the estimate $\partial_1 \tilde{p} = O(\delta_c(u_c \partial_1 \tilde{v} + \alpha \partial_1 \tilde{\tau}))$, which is valid throughout the entire critical layer. From this estimate, it follows that the terms containing $\partial_1 \tilde{p}$ in (2) have the order R^b ($b < -1$) relative to the main terms and can be rejected [if condition (9) is not satisfied, these terms reach the order $R^{-2/3}$]. System (2), (3) is parabolized in the same manner as in the case of rejected $\partial_1 \tilde{v}$. We note that an additional requirement $\Delta\xi/\xi < O(\delta_c)$ appears: the step of the marching scheme should be rather small to stay within the critical layer if condition (9) is satisfied.

Figures 1–3 show the effect of R on the spatial increment $\alpha_j = 0.5 d \ln A/dR$, where A is the amplitude of fluctuations of the mass flow [or pressure (curve 3 in Fig. 2)] for $\eta = \eta_{\max}$. All the results, including those

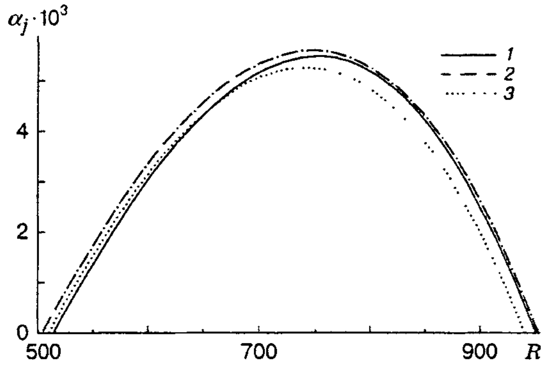

Fig. 3

taken for comparison from [1], were obtained using condition (8), except for the results shown by curves 2–4 in Fig. 2, which were calculated using condition (9). The calculations were performed for a heat-insulated wall, Pr = 1, and $\mu = T$.

The data presented in Figs. 1 and 2 were obtained for M = 5 and $\omega = 2 \cdot 10^{-4}$. The calculation results obtained using a parabolized system of equations (curve 1 in Fig. 1) predict a greater growth of perturbations than the parallel theory (curve 2), especially within the range of the fundamental mode ($R < 600$).

For calculations within the framework of the local model, we propose equations of system (6) with rejected terms containing derivatives of the amplitude functions of perturbations relative to R. The results (curve 3 in Fig. 1) are in satisfactory agreement with the results of the parabolic problem and are used in it as initial conditions.

The results for mass-flow fluctuations (curve 2 in Fig. 2) and pressure fluctuations (curve 3), which were obtained in one calculation, show how the increments of fluctuations of these parameters differ.

The fact that the results for mass-flow fluctuations do not coincide when we use condition (8) or (9) (curves 1 and 2 in Fig. 2) is explained by inadequacy (within the error of order R^{-1}) of boundary conditions (7) to system (6). Thus, Z_{ke} in (7) are functions of α, but the values of α are different for conditions (8) and (9) when the iteration process is finished. In one case, the imaginary part of α is related to α_j as $\alpha_j = -\mathrm{Im}\,(\alpha R)$ for $\tilde{m}$ (curve 1), and in the other case, for $\tilde{p}$ (curve 3). The maximum growth of the fluctuation amplitude $\ln (A_{\max}/A_{\min}) = 2 \int\limits_{\alpha_j > 0} \alpha_j \, dR$ corresponding to curves 1 and 2 is 8.1 and 8.4. For parallel and locally nonparallel theories (curves 2 and 3 in Fig. 1), these values are 5.5 and 7.4.

Curve 4 in Fig. 2 is calculated using relations (3); it is plotted to demonstrate the accuracy of Eqs. (2) and (4).

A satisfactory accuracy is reached for a step $\Delta R = 20$. Additional calculations conducted with small constant steps along ξ confirmed that restriction (5) on the step is eliminated. The calculation remains stable for $|\alpha| \Delta\xi = 0.5$, and the calculation is stable for M = 0 even if the step is reduced by a factor of 20. The results for M = 0 and $\omega = 7 \cdot 10^{-5}$ are represented by curve 1 in Fig. 3. For large R, they agree with the results of Li and Malik [1] (curve 2) and differ from the results obtained using the parallel theory (curve 3 in Fig. 3).

A comparison was performed for results obtained with and without account of terms containing $\partial_1 \tilde{p}$, and with the use of condition (9). The difference in the increments sometimes reaches 3%, and the maximum increase in amplitude is almost identical. We note that rejection of $\partial \tilde{p}/\partial \xi$ proposed previously by Li and Malik [1] for reduction of the minimum step along ξ leads to loss of accuracy if we use condition (8) or any other condition that contradicts the requirement $\tilde{p} = \mathrm{const}$ in the critical layer.

Thus, to calculate the stability of a compressible boundary layer, we recommend system (2), (4) with rejected terms that contain $\partial\tilde{p}/\partial\xi$ and with condition (9) for calculation of α. In studying the propagation of an external perturbation in the boundary layer, we propose to use system (2), (4) in the full form, since α is given and conditions like (8) and (9) are not needed. In this case, the step of the marching scheme is not limited by inequality (5), since the system does not contain $\partial\tilde{v}/\partial\xi$.

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